

Diffusions of Multiplicative Cascades

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Abstract

A multiplicative cascade can be thought of as a randomization of a measure on the boundary of a tree, constructed from an iid collection of random variables attached to the tree vertices. Given an initial measure with certain regularity properties, we construct a continuous time, measure-valued process whose value at each time is a cascade of the initial one. We do this by replacing the random variables on the vertices with independent increment processes satisfying certain moment assumptions. Our process has a Markov property: at any given time it is a cascade of the process at any earlier time by random variables that are independent of the past. It has the further advantage of being a martingale and, under certain extra conditions, it is also continuous. We discuss applications of this process to models of tree polymers and one-dimensional random geometry.

1 Introduction

Multiplicative cascades are a particular type of random measures with many interesting statistical properties. The space on which these measures live is not always the same, but there is typically a tree structure underlying their construction and so it is convenient to consider them as living on the boundary of an infinite tree. This is the situation we consider. This has the further advantage that several different models of statistical mechanics are fully described by this framework, most notably tree polymers, branching random walk, and certain models of random walk in random environment.

For simplicity we work on a rooted, infinite binary tree T , and the boundary ∂T is the set of all infinite self-avoiding paths that begin at the root. Elements of ∂T are called rays and we denote them by ξ . The inputs to the cascade model are a positive measure Γ on ∂T , which can be specified arbitrarily, and an i.i.d. collection of random variables $\{W(v)\}_{v \in T}$ attached to the vertices of the tree. The only a priori assumption on the distribution of the W is that it is strictly positive and has mean one. These random variables are then cascaded on to Γ to produce a random measure on ∂T ; we denote it by Γ_W or sometimes

$$\Gamma_W = \mathcal{C}(\Gamma; W).$$

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The cascading procedure is simple to describe: for each $n \geq 0$ one uses the random weights W up to generation n to construct a random measure via

$$d\Gamma_W^{(n)}(\xi) = \prod_{i=1}^n W(\xi_i) d\Gamma(\xi).$$

The random cascade measure is then defined as the limit

$$\Gamma_W := \lim_{n \rightarrow \infty} \Gamma_W^{(n)}. \quad (1)$$

A martingale argument shows that the limit exists almost surely for *any* choice of the initial measure Γ , in the topology of weak convergence on the space of measures. Full details are given in Section 2. As we will see there it *may* happen that Γ_W is the zero measure, but nonetheless it is well-defined, and given this the main problem is to determine the properties of Γ_W and how they depend on the input measure Γ and the cascading distribution W . Fundamental properties of cascade measures were derived in [KP76], and further explorations have been made in several later papers; see for example [Big77, HW92, LR00, OW00, Fan02].

Even in the simplest cases the relationship between Γ_W and Γ is interesting. Observe that if $W = 1$ then $\Gamma_W = \Gamma$, but if the cascading distribution is not identically one then Γ_W is necessarily distinct from Γ . There are two possible alternatives:

- Γ_W may be identically the zero measure, even though Γ is not, but
- if Γ_W is not the zero measure then it is genuinely random, meaning it depends on the specific realization of the W variables, but almost surely it is singular with respect to Γ .

The positivity of Γ_W is determined by both the regularity of Γ (roughly meaning how strongly it concentrates on some rays more than others) and moment properties of the cascading distribution. Full details are given in Section 2. The singularity property, however, holds even if the cascading distribution is highly concentrated near one. It is a simple consequence of the fact that along any ray the density is the product of positive, iid, mean one random variables, which almost surely goes to zero as the number of terms in the product goes to infinity.

The main purpose of this paper is to study what happens when the cascading distribution is highly concentrated near one and the cascading procedure is iterated. The scheme is simple: start with a positive measure Γ on ∂T and cascade once to produce Γ_W . Since the cascading procedure does not depend on the choice of the initial measure, we may use Γ_W as the input measure and cascade it with vertex variables $\{W^*(v)\}_{v \in T}$ that are independent of the $\{W(v)\}$ collection. This iteration can be repeated indefinitely, at each time cascading with a collection of vertex variables that are independent of all previous ones, and in doing so it produces a discrete time, measure-valued Markov process.

This discrete time process is interesting in its own right, but we prefer instead to study a continuous time version. Intuitively the idea behind the continuous time process is clear: starting from some initial measure, in each infinitesimal unit of time we cascade the previous measure with an independent collection of random variables whose distribution is an infinitesimal perturbation away from the degenerate distribution at one. Repeating this scheme indefinitely builds the process.

As is usual, however, rigorously constructing the continuous time process takes more care than constructing the discrete time one, even though the basic idea is the same. Several different construction techniques could be considered; for example, the discrete time process is well-defined, and the continuous time process could be constructed by taking a weak limit as the discrete time

step goes to zero and the cascading distribution concentrates near one. Alternatively, the process is essentially defined by saying that the measure at each time is a cascade of the process at an earlier time (by an independent collection of random variables); this is akin to specifying the transition probabilities of the process, and then the existence would follow from the general theory on measure-valued diffusions (see for example [EK86]).

In this paper we propose a simpler and more direct construction procedure. Instead of appealing to the more abstract concepts above, we simply attach to the vertices of the tree a family of dynamic weights $\{t \mapsto W_t(v)\}_{v \in T}$. Using the cascading procedure defined in equation (1), this gives us a process $t \mapsto \Gamma_t := \Gamma_{W_t}$ of random cascade measures. We choose the weight process $t \mapsto W_t$ so that the Γ_t process satisfies the following important Markov property: the value at any given time is a cascade of the value at any previous time, by a noise that is independent of the past of the process. More precisely, our process is defined on an interval $[0, T]$ for some $T > 0$, and has the property that for any $s, t \geq 0$ such that $t + s \leq T$, *both* of the relations

$$\Gamma_{t+s} = \mathcal{C}(\Gamma; W_{t+s}) \quad \text{and} \quad \Gamma_{t+s} = \mathcal{C}\left(\Gamma_t; \frac{W_{t+s}}{W_t}\right)$$

hold. This is a fully rigorous statement, but should be regarded as a manifestation of the non-rigorous infinitesimal cascading procedure described earlier. The main focus of our paper is to show that, under suitable assumptions on the i.i.d. collection of weight processes $W_t(v)$ attached to the vertices of the tree, the following is true:

Main Results. *Assume that the process $t \mapsto \log W_t$ is an independent increments process, with $W_0 = 1$, $\mathbf{E}[W_t] = 1$, and W_t always strictly positive. Assume the process is defined on an interval $[0, T]$ for some $T > 0$. If there is a $\delta > 0$ such that W_T has a finite $(1 + \delta)$ moment, and the measure Γ is W_T -regular (see Definition 2.1), then*

- *the process $\Gamma_t := \mathcal{C}(\Gamma, W_t)$ is well-defined on $[0, T]$, i.e. the event that $\lim_{n \rightarrow \infty} \Gamma_t^{(n)}$ exists for all $0 \leq t \leq T$ has full probability,*
- *for any $s, t \geq 0$ with $t + s \leq T$, the equality $\Gamma_{t+s} = \mathcal{C}(\Gamma_t, W_{t+s}/W_t)$ also holds almost surely,*
- *the process is a martingale with respect to the filtration $\sigma(\Gamma_s : s \leq t)$,*
- *if the process $t \mapsto W_t$ is continuous, then so is the Γ_t process in the topology of weak convergence of measures.*

These results are intuitive, but we want to emphasize that they are not immediate. It is easily seen that all four of these properties hold trivially for the finite level $t \mapsto \Gamma_t^{(n)}$ processes, but it requires some extra work to carry them over to the limit as $n \rightarrow \infty$. The main technical difficulty is that the process cannot be started from an arbitrary measure; it has to be started from those which enjoy a sufficient amount of regularity. For practical applications the regularity condition we use is not at all restrictive, but we have to ensure that once the process begins it will stay within the class of sufficiently regular measures so that it can be continued. In Section 2 we describe exactly what we mean by sufficiently regular, and in Section 3 we prove that the evolution of the regularity of the process is well-behaved. This is a part of our proof of the results above.

It is also important to note that our main technique of replacing static weights with time varying processes has already been carried out for several other models. Likely the most prominent one is Dyson's Brownian motion, which is obtained by replacing the Gaussian entries of the GUE matrices with standard Brownian motions. More recently, however, the idea has been applied to

the Sherrington-Kirkpatrick model of spin glasses in [CN95], and then re-applied to greater effect by a series of other authors [BKL02, Tin05]. The paper [MCRT11] also uses the same technique in the context of lattice polymer models, which are somewhat similar to ours through the connection between tree polymers and multiplicative cascades. However, the main purpose of these papers is to use the dynamic weights technique to derive growth exponents and fluctuation behavior for partition functions of Gibbs measures as the size of the system grows large, whereas we are more concerned with showing that the infinite volume measure-valued process has the properties listed above.

The outline of this paper is as follows: in Section 2 we set up our notation and recall some well known properties of cascade measures. In Section 3 we construct the process and show that it is well-defined, and give proofs for the main results listed above. In Section 4 we discuss the special case when the weight process is an exponential of a Brownian motion, and use stochastic calculus to describe the infinitesimal evolution of the process. This shows one advantage of our construction over the more abstract possibilities listed earlier: it allows for a full description of the evolution of the measure-valued process in terms of the input weight process $t \mapsto W_t(v)$. In Section 5 we describe possible applications of our process to models of tree polymers and to the KPZ formula of one-dimensional random geometry.

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2 Background and Notation

We begin with our notation for trees. Let T be a rooted infinite binary tree and denote the root by ς . Given a vertex $v \in T$ we let $|v|$ be its generation, by which we mean its distance from the root. Let v_L and v_R be the left and right offspring of v , respectively, and write v_p for the parent of v . We let $T(v)$ be the subtree of T rooted at v . Note that when working on subtrees we still use $|u|$ to denote the distance from ς , not from the root of the subtree.

We will mostly be working on the boundary of T , which we denote by ∂T . Recall that ∂T is the set of all infinite self-avoiding paths in the tree that begin at the root. Elements of ∂T are called rays and are usually denoted by ξ . We denote by ξ_n the vertex in the n^{th} generation of the ray ξ . Given two rays ξ and ζ we let $\xi \wedge \zeta$ be the vertex of T that is the last common ancestor of ξ and ζ . For a given vertex $v \in T$ we let $\partial T(v)$ be the set of all rays passing through v .

2.1 Measures on ∂T

Even though ∂T is an uncountable set, a measure on ∂T is completely determined by the countable collection of values $\{\Gamma(\partial T(v))\}_{v \in T}$. Hence every positive, finite measure on ∂T can be identified with a function $\Gamma : T \rightarrow \mathbb{R}_+$ satisfying the two conditions

- $0 < \Gamma(\varsigma) < \infty$,
- for every vertex $v \in T$, $\Gamma(v) = \Gamma(v_L) + \Gamma(v_R)$.

Due to this identification, measures on ∂T are also called flows on T . As long as $0 < \Gamma(\varsigma) < \infty$ it is possible to normalize Γ to be a probability measure, i.e. so that $\Gamma(\varsigma) = 1$. Sometimes we denote the normalized measure by Γ^* , but in general we do not assume that we are working with probability measures.

A special measure on ∂T is the “Lebesgue” measure given by $\theta(v) = 2^{-|v|}$. Observe that θ is also the measure induced on ∂T by constructing random paths via simple random walk; that is, starting at the root and then using independent and unbiased coin flips at each vertex to decide whether to move left or right down the tree.

The topology on measures is as follows: we say that a sequence of measures Γ_n converges to Γ if $\Gamma_n(v) \rightarrow \Gamma(v)$ for all $v \in T$. This is equivalent to weak convergence of $\Gamma_n \rightarrow \Gamma$, when the topology on ∂T is generated by the metric

$$d(\xi, \eta) = \theta(\xi \wedge \eta) = 2^{-|\xi \wedge \eta|}.$$

For a vertex $v \in T$ we will write $\Gamma|_v$ for the measure restricted to the subtree $T(v)$.

2.2 Random Cascade Measures on ∂T

In this section we describe how to take a measure Γ on ∂T and a collection of random variables to construct a cascade measure. Let W be a random variable that is positive almost surely and has mean one. We are mostly concerned with its distribution which we call the *cascading distribution*. Assume that W is not identically one, and therefore Jensen’s inequality implies that $\mathbf{E}[\log W] < 0$. Later we may make additional assumptions on the distribution of W , but for now this is all that we require.

Let $\{W(v)\}_{v \in T}$ be a collection of i.i.d. random variables with common distribution W . From this collection we build a random function $X : T \rightarrow \mathbb{R}_+$ defined by

$$X(\xi_n) = \prod_{i=1}^n W(\xi_i).$$

Then for each $n \geq 0$ we construct a random measure $\Gamma_W^{(n)}$ by specifying the Radon-Nikodym derivative with respect to Γ as

$$d\Gamma_W^{(n)}(\xi) := X(\xi_n) d\Gamma(\xi).$$

The random cascade measure is then defined as the limit of $\Gamma_W^{(n)}$ as $n \rightarrow \infty$. Recall that the topology is pointwise in the vertices, meaning that

$$\Gamma_W(v) = \lim_{n \rightarrow \infty} \Gamma_W^{(n)}(v) \tag{2}$$

for every $v \in T$. A simple martingale argument, which we now recall, shows that the limit always exists. First consider the case $v = \varsigma$, so that

$$\Gamma_W^{(n)}(\varsigma) = \int_{\partial T} X(\xi_n) d\Gamma(\xi).$$

It is easy to see that $X(\xi_n)$ is a martingale with respect to the filtration

$$\mathcal{W}_n := \sigma(W(v) : |v| \leq n),$$

and therefore so is $\Gamma_W^{(n)}(\varsigma)$ by Fubini’s Theorem. Since $\Gamma_W^{(n)}(\varsigma)$ is positive it converges almost surely, and since positivity of the limit does not depend on any finite collection of the $W(v)$ variables

a standard 0-1 law argument shows that the limit is almost surely zero or almost surely strictly positive. In the case that $\Gamma = \theta$ Kahane and Peyriere [KP76] showed that

$$\mathbf{E}[W \log W] < \log 2$$

is a necessary and sufficient condition for the limit to be positive. In the case of a general measure Γ it remains an open problem to determine sharp criterion for when $\Gamma_W(\varsigma) > 0$, but there are many known sufficient conditions involving moment of W and the regularity of Γ . We will use a condition of Fan [Fan02], for which we need the following definitions:

Definition 2.1. For Γ a measure on ∂T , define the *pressure function* $\lambda_\Gamma : [0, \infty) \rightarrow \mathbb{R}$ by

$$\lambda_\Gamma(h) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{|v|=n} \Gamma(v)^h.$$

We will say that a measure Γ on ∂T is W -regular if

$$\mathbf{E}[W \log W] + \lambda'_\Gamma(1+) < 0.$$

We say that it is W -irregular if

$$\mathbf{E}[W \log W] + \lambda'_\Gamma(1-) > 0.$$

Observe that $\lambda_\Gamma(1) = 0$ for any Γ . Fan [Fan02] uses the pressure function to derive the following condition:

Proposition 2.2 ([Fan02]). *Suppose there exists a $\delta > 0$ with $\mathbf{E}[W^{1+\delta}] < \infty$ for some $\delta > 0$. Then*

- *if Γ is W -regular then $\Gamma_W(\varsigma) > 0$ almost surely,*
- *if Γ is W -irregular then $\Gamma_W(\varsigma) = 0$ almost surely.*

Observe that if λ_Γ is differentiable at $h = 1$ then the condition of W -regularity is close to sharp. For $\Gamma = \theta$ we have $\lambda_\theta(h) = (1-h) \log 2$, and hence the condition of Kahane and Peyriere is recovered.

Remark 1. Let W_1 and W_2 be two distinct cascading distributions, and suppose there is an $\epsilon > 0$ such that $\mathbf{E}[W_1^h] \leq \mathbf{E}[W_2^h] < \infty$ for $h \in [1, 1+\epsilon]$. Then since

$$\mathbf{E}[W \log W] = \lim_{h \downarrow 0} \frac{\mathbf{E}[W^h] - 1}{h}$$

it follows that $\mathbf{E}[W_1 \log W_1] \leq \mathbf{E}[W_2 \log W_2]$. Hence W_2 -regularity of Γ implies W_1 -regularity of Γ .

Remark 2. Assume that $\mathbf{E}[W^{1+\delta}] < \infty$ for some $\delta > 0$, and define

$$\alpha(h) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{|v|=n} \Gamma(v)^h \mathbf{E}[X(v)^h] = \lambda_\Gamma(h) + \log \mathbf{E}[W^h].$$

The moment assumption on W implies that $\alpha(h) < \infty$ for h in a neighborhood of 1. Since $\alpha(1) = 0$ it is straightforward to compute that Γ being W -regular implies that $\alpha'(1+) < 0$, and therefore $\alpha(1+\epsilon) < \alpha(1) = 0$ for ϵ sufficiently small. We will use this fact repeatedly in Section 3.

Remark 3. The assumption of W -regularity implicitly means that λ_Γ is differentiable from the right at $h = 1$.

Remark 4. It is important to note that W -regularity of a measure is a property that is inherited by all of its submeasures. Indeed, since λ_Γ is computed over a larger sum than $\lambda_{\Gamma|_v}$, it follows that $\lambda_\Gamma(h) \geq \lambda_{\Gamma|_v}(h)$ for all h . But also $\lambda_\Gamma(1) = \lambda_{\Gamma|_v}(1) = 0$, and therefore by the Mean Value Theorem

$$0 \leq \lambda_\Gamma(1 + \epsilon) - \lambda_{\Gamma|_v}(1 + \epsilon) = \lambda'_\Gamma(1 + s) - \lambda'_{\Gamma|_v}(1 + s)$$

for some $s \in (0, \epsilon)$. Taking ϵ to zero gives

$$\lambda'_{\Gamma|_v}(1+) \leq \lambda'_\Gamma(1+),$$

which implies W -regularity of $\Gamma|_v$.

Remark 5. We have only shown existence of the limit (2) in the $v = \varsigma$ case, and it is important to note that this only required that Γ is a positive measure on ∂T . The regularity of Γ determines whether the limit is positive or zero. But these facts and the self-similarity of the tree also combine to give us the existence and positivity of the limit for $v \neq \varsigma$. Indeed, assume $n > |v|$, so that

$$\begin{aligned} \Gamma_W^{(n)}(v) &= \int_{\partial T} X(\xi_n) \mathbf{1}_{\{\xi \in \partial T(v)\}} d\Gamma(\xi) \\ &= X(v) \int_{\partial T(v)} \frac{X(\xi_{n-|v|})}{X(v)} d\Gamma|_v(\xi). \end{aligned} \tag{3}$$

But the integral term is just the level $n - |v|$ cascade of the $\Gamma|_v$ measure by the random variables $W|_v = \{W(u) : u \in T(v)\}$, hence the martingale argument for the $v = \rho$ case also shows that its limit exists as $n \rightarrow \infty$. Its positivity can again be determined by Fan's condition, and by the last remark W -regularity is inherited by all submeasures. Thus if Γ is W -regular then $\Gamma_W(v) > 0$ for all $v \in T$ with $\Gamma(v) > 0$. Taking the limit as $n \rightarrow \infty$ in equation (3) gives the relation

$$\frac{\Gamma_W(u)}{X(v)} = \mathcal{C}(\Gamma|_v; W|_v)(u) \tag{4}$$

for all $u \in T(v)$.

Finally we remark that even though the limits in (2) are defined pointwise at each vertex, the limiting object Γ_W is automatically a measure on ∂T . This follows from the definition of the level n cascade as a measure, and therefore

$$\Gamma_W^{(n)}(v) = \Gamma_W^{(n)}(v_L) + \Gamma_W^{(n)}(v_R).$$

Now take limits as $n \rightarrow \infty$.

2.3 Rates of Convergence for the Cascading Procedure

Much of our analysis will rely on having a rate of convergence of $\Gamma_W^{(n)}$ to Γ_W . We will heavily make use of the following lemma:

Lemma 2.3. For $1 \leq h \leq 2$ there is a positive constant $C = C(h)$ such that

$$\left\| \Gamma_W^{(n+1)}(\varsigma) - \Gamma_W^{(n)}(\varsigma) \right\|_{L^h} \leq C \|W\|_{L^h}^{n+1} \left(\sum_{|v|=n} \Gamma(v)^h \right)^{1/h},$$

and therefore by the triangle inequality

$$\left\| \Gamma_W(\varsigma) - \Gamma_W^{(n)}(\varsigma) \right\|_{L^h} \leq C \sum_{m>n} \|W\|_{L^h}^m \left(\sum_{|v|=m} \Gamma(v)^h \right)^{1/h}.$$

The proof relies on the following inequality of von Bahr and Esseen:

Lemma 2.4 ([vBE65]). Let $\{U_i\}$ and $\{V_i\}$ be sequences of random variables that are independent of each other. Also assume that the $\{V_i\}$ are mutually independent, and that $\mathbf{E}[V_i] = 0$ for all i . Then for $1 \leq h \leq 2$ there is a universal constant $c = c(h)$ such that

$$\mathbf{E} \left[\left(\sum_i U_i V_i \right)^h \right] \leq c \sum_i \mathbf{E} [U_i^h] \mathbf{E} [V_i^h].$$

Proof of Lemma 2.3. We have the trivial identity

$$\begin{aligned} \Gamma^{(n+1)}(\varsigma) - \Gamma^{(n)}(\varsigma) &= \int (X(\xi_{m+1}) - X(\xi_m)) d\Gamma(\xi) \\ &= \int X(\xi_m)(W(\xi_{m+1}) - 1) d\Gamma(\xi) \\ &= \sum_{|v|=m+1} \Gamma(v) X(v_p)(W(v) - 1). \end{aligned}$$

The von Bahr-Esseen inequality applies to the latter sum, and therefore

$$\begin{aligned} \mathbf{E} \left[\left| \Gamma_W^{(n+1)}(\varsigma) - \Gamma_W^{(n)}(\varsigma) \right|^h \right] &\leq c(h) \sum_{|v|=n+1} \Gamma(v)^h \mathbf{E} [W^h]^n \mathbf{E} [|W - 1|^h] \\ &\leq 2c(h) \mathbf{E} [W^h]^{n+1} \sum_{|v|=n+1} \Gamma(v)^h. \end{aligned}$$

□

The next lemma implies the L^h convergence of the total mass of the cascade measure, and therefore that $\Gamma_W(\varsigma) > 0$ almost surely.

Corollary 2.5. Suppose there is a $\delta > 0$ such that $\mathbf{E}[W^{1+\delta}] < \infty$. If Γ is W -regular, then for all $h = 1 + \epsilon$ with ϵ sufficiently small,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E} \left[|\Gamma_W(\varsigma) - \Gamma_W^{(n)}(\varsigma)|^h \right] \leq \lambda_\Gamma(h) + \log \mathbf{E}[W^h] < 0.$$

Proof. Since Γ is W -regular, by Remark 2 for all $h = 1 + \epsilon$ with ϵ sufficiently small we have $\alpha(h) = \lambda_\Gamma(h) + \log \mathbf{E}[W^h] < 0$. Hence for each $\gamma > 0$ such that $\alpha(h) + \gamma < 0$ there is a positive constant C such that

$$\sum_{|v|=n} \Gamma(v)^h \mathbf{E}[X(v)^h] \leq C e^{n(\alpha(h)+\gamma)}$$

for all n . Applying the second statement of Lemma 2.3 completes the proof. \square

We now extend Corollary 2.5 to show that the exponential rate of convergence is uniform for all vertices on a fixed generation of the tree.

Lemma 2.6. *Suppose $\mathbf{E}[W^{1+\delta}] < \infty$ for some $\delta > 0$ and that Γ is W -regular. Then for $1 \leq i \leq n$*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{|v|=i} \mathbf{E} \left[\left| \Gamma_W(v) - \Gamma_W^{(n)}(v) \right|^h \right] \leq \lambda_\Gamma(h) + \log \mathbf{E}[W^h] < 0$$

for all $h = 1 + \epsilon$ with ϵ sufficiently small. Moreover

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{i=1}^n \sum_{|v|=i} \mathbf{E} \left[\left| \Gamma_W(v) - \Gamma_W^{(n)}(v) \right|^h \right] \leq \lambda_\Gamma(h) + \log \mathbf{E}[W^h] < 0$$

for a similar range of h .

Proof. From (3) we have, for $|v| = i \leq n$,

$$\Gamma_W^{(n)}(v) = X(v) \mathcal{C}(\Gamma_{|v}; W_{|v})^{(n-i)}(v).$$

Combining this with (4) and using that $X(v)$ is independent of the cascade on $T(v)$ gives

$$\mathbf{E} \left[\left| \Gamma_W(v) - \Gamma_W^{(n)}(v) \right|^h \right] = \mathbf{E} \left[X(v)^h \right] \mathbf{E} \left[\left| \mathcal{C}(\Gamma_{|v}; W_{|v})(v) - \mathcal{C}(\Gamma_{|v}; W_{|v})^{(n-i)}(v) \right|^h \right].$$

Applying Lemma 2.3 to the second factor and combining with the first factor gives

$$\mathbf{E} \left[\left| \Gamma_W(v) - \Gamma_W^{(n)}(v) \right|^h \right] \leq C \left(\sum_{k > n-i} \left(\sum_{\substack{w \in T(v) \\ |w|=|v|+k}} \Gamma(w)^h \mathbf{E}[W^h]^{|v|+k} \right)^{1/h} \right)^h, \quad (5)$$

where C depends only on h . Now define $a_k(v)$ by

$$a_k(v) = \sum_{\substack{w \in T(v) \\ |w|=|v|+k}} \Gamma(w)^h \mathbf{E}[W^h]^{|v|+k}$$

and $\mathbf{a}_n(v) = (a_{n+1}(v), a_{n+2}(v), a_{n+3}(v), \dots)$. Then equation (5) is equivalent to

$$\mathbf{E} \left[\left| \Gamma_W(v) - \Gamma_W^{(n)}(v) \right|^h \right] \leq C \|\mathbf{a}_{n-i}(v)\|_{\ell^{1/h}},$$

with $\ell^{1/h}$ denoting the usual sequence space. Summing over $|v| = i$ gives

$$\sum_{|v|=i} \mathbf{E} \left[|\Gamma_W(v) - \Gamma_W^{(n)}(v)|^h \right] \leq C \sum_{|v|=i} \|\mathbf{a}_{\mathbf{n}-\mathbf{i}}(v)\|_{\ell^{1/h}} \leq C \left\| \sum_{|v|=i} \mathbf{a}_{\mathbf{n}-\mathbf{i}}(v) \right\|_{\ell^{1/h}}. \quad (6)$$

The last inequality is the Minkowski inequality for $\ell^{1/h}$ (recall $h \geq 1$). By definition of \mathbf{a} we have

$$\sum_{|v|=i} \mathbf{a}_{\mathbf{n}-\mathbf{i}}(v) = (a_{n+1}(\varsigma), a_{n+2}(\varsigma), a_{n+3}(\varsigma), \dots) = \mathbf{a}_{\mathbf{n}}(\varsigma). \quad (7)$$

Now define $\alpha(h) = \lambda_\Gamma(h) + \log \mathbf{E} [W^h]$. By definition of $\alpha(h)$ we have, for each $\gamma > 0$,

$$a_n(\varsigma) = \sum_{|v|=n} \Gamma(v)^h \mathbf{E} [W^h]^n \leq e^{n(\alpha(h)+\gamma)}$$

for n sufficiently large. As we saw in Remark 2, the $(1 + \delta)$ moment assumption on W and the W -regularity of Γ imply that $\alpha(h) < 0$, for $h = 1 + \epsilon$ with ϵ sufficiently small. Choosing γ such that $\alpha(h) + \gamma < 0$, this gives

$$\|\mathbf{a}_{\mathbf{n}}(\varsigma)\|_{\ell^{1/h}} \leq C e^{n(\alpha(h)+\gamma)}$$

for n sufficiently large. Combining this with (6) and (7) and sending γ to zero gives the first statement of the lemma. For the second part, simply observe that by (6) we have

$$\sum_{i=1}^n \sum_{|v|=i} \mathbf{E} \left[|\Gamma_W(v) - \Gamma_W^{(n)}(v)|^h \right] \leq C n \|\mathbf{a}_{\mathbf{n}}(\varsigma)\|_{\ell^{1/h}}.$$

□

3 A Markovian Random Cascade Process

3.1 Dynamic Random Weights

The main idea of this paper is to replace the random weights on the vertices of the tree with random weight processes that evolve in time. Throughout we assume that our weight processes have the following properties:

Assumption 1. The weight process $t \mapsto W_t$ satisfies

- $W_0 = 1$,
- $W_t > 0$ and $\mathbf{E}[W_t] = 1$ for each $t \geq 0$,
- $t \mapsto \log W_t$ has independent increments.

In later sections we will assume more properties, but for now this is all that we require. Such processes are easy to construct, for example exponentials of Brownian motion or exponentials of Levy processes (both properly normalized so that $\mathbf{E}[W_t] = 1$). Note, however, that in both of

these examples the increments of $\log W_t$ are stationary, but that our results do *not* require this. For $s, t \geq 0$ we define

$$W_{t,t+s} := \frac{W_{t+s}}{W_t}.$$

The independent increments assumption gives that $W_{t,t+s}$ is independent of W_t . Moreover the process $t \mapsto W_t$ is a martingale, that is

$$\mathbf{E}[W_t | \sigma(W_r : r \leq s)] = W_s.$$

Now to each vertex $v \in T$ attach a copy $W_t(v)$ of this process such that the collection $\{W_t(v)\}_{v \in T}$ is independent. The main idea of this paper is to use the cascading procedure of the last section to construct a random cascade measure Γ_{W_t} at each time $t \geq 0$, and then show that the resulting process $t \mapsto \Gamma_{W_t}$ is Markovian. This is carried out in Section 3.3, and the rest of the paper studies properties of the process. To simplify notation we write

$$\Gamma_t := \Gamma_{W_t} = \mathcal{C}(\Gamma; W_t).$$

Observe that $\Gamma_0 = \Gamma$. We also define functions $X_{t,t+s} : T \rightarrow \mathbb{R}_+$ by

$$X_{t,t+s}(\xi_n) = \prod_{i=1}^n W_{t,t+s}(\xi_i),$$

and the filtrations

$$\mathcal{W}_t = \sigma(W_s(v) : v \in T, s \leq t), \quad \mathcal{F}_t = \sigma(\Gamma_s(v) : v \in T, s \leq t).$$

In general $\mathcal{F}_t \subset \mathcal{W}_t$ and the inclusion is strict, since by knowing the weights one can construct the measure, but knowing the measure does not generally give full information on the weights.

In applying the weight process to cascade from some initial measure Γ we make the following additional assumptions:

Assumption 2. The weight process $t \mapsto W_t$ is defined in an interval $[0, T]$ with $T > 0$, and

- there is a $\delta > 0$ such that $\mathbf{E}[W_T^{1+\delta}] < \infty$,
- the measure Γ is W_T -regular.

Remark 6. Observe that for $p > 1$

$$\mathbf{E}[W_T^p] = \mathbf{E}[W_t^p] \mathbf{E}[W_{t,T}^p] \geq \mathbf{E}[W_t^p],$$

and so, by Remark 1, Assumptions 2 are inherited for W_t with $t < T$.

3.2 Construction and Basic Properties of the Process

Before proving that the $t \mapsto \Gamma_t$ process is Markov we first deal with a technical issue. Above we said that we construct the process $t \mapsto \Gamma_t$ by applying the random cascading procedure of Section 2 at each fixed time t . However the existence of the random cascade measure is only an almost sure statement, and the event that it does not exist could conceivably depend on t . Since we are now working in continuous time it is possible that there is an exceptional set of times for which

the cascade does *not* exist, which would leave our cascade process ill-defined. We begin by showing that this is not the case.

To this end first note that for each $n > 0$ the finite level measure processes $t \mapsto \Gamma_t^{(n)}$ are well-defined, and in fact are martingales in t with respect to the filtration \mathcal{W}_t . Indeed

$$\mathbf{E} \left[d\Gamma_{t+s}^{(n)}(\xi) | \mathcal{W}_t \right] = X_t(\xi_n) d\Gamma(\xi) \mathbf{E} [X_{t,t+s}(\xi_n)] = d\Gamma_t^{(n)}(\xi),$$

by the fact that $X_{t,t+s}$ is independent of \mathcal{W}_t and has mean one. We will show that this martingale property, together with the exponential L^p convergence of the finite level measures, gives that the Γ_t process is well-defined. Moreover, the martingale property of the finite level measures is inherited by the limit.

Theorem 3.1. *Under Assumptions 1 and 2, the event*

$$\left\{ \lim_{n \rightarrow \infty} \Gamma_t^{(n)}(v) \text{ exists for all } v \in T, t \leq T \right\}$$

has probability one. Moreover,

- (i) *for each $v \in T$ the process $t \mapsto \Gamma_t(v)$ is a martingale with respect to \mathcal{W}_t , and hence \mathcal{F}_t , and,*
- (ii) *if the weight process $t \mapsto W_t$ is continuous then so is $\Gamma_t(v)$ for each $v \in T$.*

Proof. We concentrate first on the case $v = \varsigma$. Since the difference $\Gamma_t^{(n+1)}(\varsigma) - \Gamma_t^{(n)}(\varsigma)$ is a martingale in t (with respect to the filtration \mathcal{W}_t), we may apply Doob's maximal L^h inequality to get that

$$\begin{aligned} \mathbb{P} \left(\sup_{0 \leq t \leq T} |\Gamma_t^{(n+1)}(\varsigma) - \Gamma_t^{(n)}(\varsigma)| > \beta^n \right) &\leq \beta^{-nh} \mathbf{E} \left[\sup_{0 \leq t \leq T} |\Gamma_t^{(n+1)}(\varsigma) - \Gamma_t^{(n)}(\varsigma)|^h \right] \\ &\leq \beta^{-nh} \left(\frac{h}{h-1} \right)^h \mathbf{E} \left[|\Gamma_T^{(n+1)}(\varsigma) - \Gamma_T^{(n)}(\varsigma)|^h \right] \\ &\leq C \beta^{-nh} \mathbf{E} \left[W_T^h \right]^n \sum_{|v|=n} \Gamma(v)^h, \end{aligned}$$

with the last inequality coming from Lemma 2.3.

Now by Assumption 2 and Remark 2, there is an $h > 1$ and an $\alpha < 1$ such that

$$\mathbf{E} \left[W_T^h \right]^n \sum_{|v|=n} \Gamma(v)^h < \alpha^n$$

for n sufficiently large. Thus by the last lemma

$$\mathbb{P} \left(\sup_{0 \leq t \leq T} |\Gamma_t^{(n+1)}(\varsigma) - \Gamma_t^{(n)}(\varsigma)| > \beta^n \right) \leq C \beta^{-nh} \alpha^n,$$

where the constant C only depends on h . Now choose β such that $\alpha^{1/h} < \beta < 1$ and apply Borel-Cantelli to conclude that $\Gamma_t^{(n)}(\varsigma)$ is a Cauchy sequence in n , with a rate of convergence that is uniform in t . This proves the first part of the theorem.

To prove the martingale property, simply note that by Corollary 2.5 there is an $h > 1$ such that $\Gamma_t^{(n)}(\varsigma)$ converges to $\Gamma_t(\varsigma)$ in L^h , and hence in L^1 . Thus for $A \in \mathcal{W}_s$

$$\mathbf{E} [(\Gamma_{t+s}(\varsigma) - \Gamma_t(\varsigma)) \mathbf{1}_A] = \lim_{n \rightarrow \infty} \mathbf{E} [(\Gamma_{t+s}^{(n)}(\varsigma) - \Gamma_t^{(n)}(\varsigma)) \mathbf{1}_A] = 0,$$

with the last equality using the martingale property of the finite level measure process. This proves that Γ_t is a martingale with respect to \mathcal{W}_t , but since $\mathcal{F}_t \subset \mathcal{W}_t$ and Γ_t is \mathcal{F}_t -measurable, it is automatically a martingale with respect to \mathcal{F}_t also.

For the continuity statement observe that if W_t is continuous then so is $\Gamma_t^{(n)}(\varsigma)$, since it is a finite product and sum of continuous functions. The Borel-Cantelli argument above gives continuity of $\Gamma_t(\varsigma)$ by completeness of $C([0, T])$ under the sup norm.

Finally, if $v \neq \varsigma$ then the proofs above are easily extended by noting that W_T -regularity is inherited by the submeasures $\Gamma|_v$ (see Remark 4). The simple relation $\Gamma_t(v) = X_t(v)\mathcal{C}(\Gamma|_v, W_t)(v)$ finishes the argument, and since there are only countably many vertices on the tree the proof is completed. \square

3.3 The Markov Property

In this section we show that the Γ_t process has the Markov property. For a given weight process W_t on $[0, T]$, let \mathcal{M}_T be the space of measures Γ that satisfy Assumption 2. The Markov property can be formally stated by saying that for any bounded, measurable $F : \mathcal{M}_T \rightarrow \mathbb{R}$ we have

$$\mathbf{E}[F(\Gamma_{t+s}) | \mathcal{F}_t] = \mathbf{E}[F(\Gamma_{t+s}) | \Gamma_t],$$

for $s, t \geq 0$ such that $s + t \leq T$. By a density argument it is sufficient to consider the functions of the form $F_v(\Gamma) = \Gamma(v)$ for $v \in T$. For these functions we will actually prove the stronger statement

$$\mathbf{E}[F_v(\Gamma_{t+s}) | \mathcal{W}_t] = \mathbf{E}[F_v(\Gamma_{t+s}) | \Gamma_t],$$

the difference between the two being that \mathcal{W}_t is a coarser σ -algebra than \mathcal{F}_t . Since the weight processes $s \mapsto W_{t+s}$ are independent of \mathcal{W}_t , it is sufficient to prove the following:

Theorem 3.2. *Under Assumptions 1 and 2, for fixed $s, t \geq 0$ such that $t + s \leq T$, we have with probability one that*

$$\Gamma_{t+s} = \mathcal{C}(\Gamma_t; W_{t,t+s}).$$

Proof. We will show that for every v in T ,

$$\Gamma_{t+s}(v) = \mathcal{C}(\Gamma_t; W_{t,t+s})(v). \quad (8)$$

We first concentrate on the case $v = \varsigma$. Note that both sides of equation (8) are defined as limits, and it suffices to prove that

$$\lim_{n \rightarrow \infty} \mathcal{C}(\Gamma_t; X_{t,t+s})^{(n)}(\varsigma) - \Gamma_{t+s}^{(n)}(\varsigma) = 0. \quad (9)$$

We will show that the left hand side of (9) goes to zero in L^h for some $h > 1$, and therefore the a.s. limit must be zero as well. Recall that

$$\mathcal{C}(\Gamma_t; X_{t,t+s})^{(n)}(\varsigma) = \sum_{|v|=n} \Gamma_t(v) X_{t,t+s}(v) = \sum_{|v|=n} \frac{\Gamma_t(v)}{X_t(v)} X_{t+s}(v).$$

The last equality follows since $X_t X_{t,t+s} = X_{t+s}$ by construction. Therefore, by definition of $\Gamma_{t+s}^{(n)}$,

$$\mathcal{C}(\Gamma_t; X_{t,t+s})^{(n)}(\varsigma) - \Gamma_{t+s}^{(n)}(\varsigma) = \sum_{|v|=n} \left(\frac{\Gamma_t(v)}{X_t(v)} - \Gamma(v) \right) X_{t+s}(v).$$

Now note that the random variables $\{\Gamma_t(v)/X_t(v) - \Gamma(v) : |v| = n\}$ are mean zero, and each depends only on the W_t weights in the subtree $T(v)$. Hence they are independent of each other *and* of all the weight processes $t \mapsto W_t(v)$ with $|v| \leq n$. In particular each $X_{t+s}(v)$, for $|v| = n$, is independent of these random variables. Thus we can apply the von Bahr-Esseen inequality to the difference above to get

$$\begin{aligned} \mathbf{E} \left[\left| \mathcal{C}(\Gamma_t; X_{t,t+s})^{(n)}(\varsigma) - \Gamma_{t+s}^{(n)}(\varsigma) \right|^h \right] &\leq \sum_{|v|=n} \mathbf{E} \left[X_{t+s}(v)^h \right] \mathbf{E} \left[\left| \frac{\Gamma_t(v)}{X_t(v)} - \Gamma(v) \right|^h \right] \\ &= \sum_{|v|=n} \mathbf{E} \left[X_{t,t+s}(v)^h \right] \mathbf{E} \left[|\Gamma_t(v) - X_t(v)\Gamma(v)|^h \right]. \end{aligned}$$

Recognizing that $X_t(v)\Gamma(v) = \Gamma_t^{(n)}(v)$ and applying Lemma 2.6 finishes the proof, since for some $h > 1$ we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbf{E} \left[\left| \mathcal{C}(\Gamma_t; X_{t,t+s})^{(n)}(\varsigma) - \Gamma_{t+s}^{(n)}(\varsigma) \right|^h \right] &\leq \log \mathbf{E} \left[W_{t,t+s}^h \right] + \lambda_\Gamma(h) + \log \mathbf{E} \left[W_t^h \right] \\ &= \lambda_\Gamma(h) + \log \mathbf{E} \left[W_{t+s}^h \right] \\ &\leq \lambda_\Gamma(h) + \log \mathbf{E} \left[W_T^h \right] \\ &< 0. \end{aligned}$$

The second inequality follows from Remark 6, and the last is by the W_T -regularity of Γ .

The proof for $v \neq \varsigma$ is similar, with all sums in the above statements being replaced with sums over the appropriate subtrees, and by making use of the fact that $\Gamma|_v$ is W_T -regular for all v . Finally, since there are only countably many vertices on the tree the statement holds for all vertices simultaneously. \square

Note that Theorem 3.2 assumes nothing about the regularity of Γ_t , even though we applied the cascading procedure to it. Theorem 3.2 gives that $\mathcal{C}(\Gamma_t, W_{t,t+s})$ is indeed a non-trivial measure since it is equal to Γ_{t+s} , which was already known to be non-trivial by the W_T -regularity of the original measure Γ . However, the regularity condition is only a sufficient one, and so the fact that $\mathcal{C}(\Gamma_t, W_{t,t+s})$ is non-trivial does not imply that Γ_t is $W_{t,T}$ regular. This regularity statement is true, however, and we will now prove it. In some sense this gives a classification of the state space of the Markov process: each Γ_t lives in the space of $W_{t,T}$ -regular measures, which is itself contained in the space of W_T -regular measures.

Lemma 3.3. *Under Assumptions 1 and 2, the measures Γ_t are $W_{t,T}$ -regular for each $t \leq T$.*

Proof. From the inequality $|a + b|^h \leq 2^h(|a|^h + |b|^h)$ we have

$$\mathbf{E} \left[\Gamma_t(v)^h \right] \leq 2^h \left(\mathbf{E} \left[|\Gamma_t(v) - X_t(v)\Gamma(v)|^h \right] + \Gamma(v)^h \mathbf{E} \left[X_t(v)^h \right] \right),$$

and then summing over $|v| = n$ and taking logarithms we get

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{|v|=n} \mathbf{E} \left[\Gamma_t(v)^h \right] &\leq \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{|v|=n} \left(\mathbf{E} \left[|\Gamma_t(v) - X_t(v)\Gamma(v)|^h \right] + \Gamma(v)^h \mathbf{E} \left[X_t(v)^h \right] \right) \\ &\leq \lambda_\Gamma(h) + \log \mathbf{E} \left[W_t^h \right]. \end{aligned}$$

The last inequality is a consequence of the fact that the two terms in the line above both have the same exponential rate of decay, which is itself a consequence of Lemma 2.6. We also have that $\lambda_\Gamma(h) + \log \mathbf{E} [W_t^h] < 0$ for $h \in [1, 1 + \epsilon]$ with ϵ sufficiently small, by the assumption that Γ is W_T -regular (and hence W_t -regular), and therefore an application of Borel-Cantelli implies that

$$\lambda_{\Gamma_t}(h) = \limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{|v|=n} \Gamma_t(v)^h \leq \lambda_\Gamma(h) + \log \mathbf{E} [W_t^h]$$

with probability one. This gives that

$$\lambda_{\Gamma_t}(h) + \log \mathbf{E} [W_{t,T}^h] \leq \lambda_\Gamma(h) + \log \mathbf{E} [W_t^h] + \log \mathbf{E} [W_{t,T}^h] = \lambda_\Gamma(h) + \log \mathbf{E} [W_T^h]$$

for all $h \in [1, 1 + \epsilon]$. Now apply the Mean Value Theorem and take $h \downarrow 1$ to finish the proof. \square

4 Gaussian Weight Processes

The simplest case of weights satisfying Assumption 1 is an exponential of a Brownian motion, properly normalized. In this section we study some extra properties of the random cascade process with these weights; specifically we derive stochastic calculus formulas for the evolution of the measures as driven by the Brownian noise. We restrict ourselves to the simplest case when $t \mapsto \log W_t$ has stationary increments, so that

$$W_t(v) = \exp \{B_t(v) - t/2\},$$

where $\{B_t(v)\}_{v \in T}$ is a collection of independent Brownian motions with $B_0(v) = 0$. Since the W_t variables have moments of all orders for all $t \geq 0$, we only need to assume that the initial measure Γ is W_T -regular for some $T > 0$. It is easy to compute that

$$\mathbf{E} [W_t \log W_t] = -\frac{t}{2},$$

so therefore the cascade process is well-defined on $[0, -2\lambda'_\Gamma(1+))$. It is straightforward to verify from the definition of λ_Γ that $-2\lambda'_\Gamma(1+)$ is maximal when $\Gamma = \theta$, and this maximum value is $2 \log 2$. Hence the $t \mapsto \Gamma_t$ process is always defined on a finite time interval. By Theorems 3.1 and 3.2 the process is Markovian, and $t \mapsto \Gamma_t(v)$ is a continuous martingale for each $v \in T$. Since

$$X_t(\xi_n) = \exp \left\{ \sum_{i=1}^n B_t(\xi_i) - nt/2 \right\},$$

it is easy to compute that

$$\frac{dX_t(\xi_n)}{X_t(\xi_n)} = \sum_{i=1}^n dB_t(\xi_i).$$

Therefore

$$d\Gamma_t^{(n)}(\varsigma) = \int_{\partial T} X_t(\xi_n) \left(\sum_{i=1}^n dB_t(\xi_i) \right) d\Gamma(\xi) = \sum_{i=1}^n \int_{\partial T} dB_t(\xi_i) d\Gamma_t^{(n)}(\xi). \quad (10)$$

This leads to the following result:

Proposition 4.1. *The total mass $\Gamma_t(\varsigma)$ evolves according to the stochastic differential equation*

$$d\Gamma_t(\varsigma) = \sum_{i=1}^{\infty} \int_{\partial T} dB_t(\xi_i) d\Gamma_t(\xi) = \sum_{i=1}^{\infty} \mathbf{E}_{\Gamma_t} [dB_t(\xi_i)] = \sum_{\substack{v \in T \\ v \neq \varsigma}} \Gamma_t(v) dB_t(v), \quad (11)$$

where all stochastic differentials are understood in the Itô sense. Equivalently

$$\frac{d\Gamma_t(\varsigma)}{\Gamma_t(\varsigma)} = \sum_{i=1}^{\infty} \mathbf{E}_{\Gamma_t^*} [dB_t(\xi_i)] = \sum_{\substack{v \in T \\ v \neq \varsigma}} \Gamma_t^*(v) dB_t(v),$$

where Γ_t^* is Γ_t normalized to be a probability measure. The quadratic variation of the latter process is

$$\frac{d \langle \Gamma_t(v), \Gamma_t(v) \rangle}{\Gamma_t(v)^2} = \sum_{i=1}^{\infty} \mathbf{E}_{\Gamma_t^* \times \Gamma_t^*} [\mathbf{1} \{ \xi_i = \xi'_i \}] = \sum_{\substack{v \in T \\ v \neq \varsigma}} \Gamma_t^*(v)^2 = \sum_{\substack{v \in T \\ v \neq \varsigma}} \left(\frac{\Gamma_t(v)}{\Gamma_t(\varsigma)} \right)^2.$$

Before proceeding with the proof we first note that all of the stochastic integrals

$$\int_0^t \Gamma_s^{(n)}(v) dB_s(v), \quad \int_0^t \Gamma_s(v) dB_s(v)$$

are well-defined on $[0, T]$. Both integrands are clearly progressively measurable, and as they are continuous local martingales in time their supremum is almost surely finite on the compact interval $[0, T]$. Hence

$$\int_0^T \Gamma_s^{(n)}(v)^2 ds < \infty \quad \text{and} \quad \int_0^T \Gamma_s(v)^2 ds < \infty$$

with probability one, which is exactly what is required for the integrals to make sense. Note, however, that the expectations of the latter integrals will not necessarily be finite for all T .

Proof. By the definition of $\Gamma_t(\varsigma)$ as the limit of $\Gamma_t^{(n)}(\varsigma)$, and computing the difference between (10) and (11), it is sufficient to show that the process

$$t \mapsto \sum_{i=1}^n \sum_{|v|=i} \int_0^t \left(\Gamma_s(v) - \Gamma_s^{(n)}(v) \right) dB_s(v) + \sum_{i=n+1}^{\infty} \sum_{|v|=i} \int_0^t \Gamma_s(v) dB_s(v) \quad (12)$$

goes to zero in some sense as $n \rightarrow \infty$. We will show that the supremum over $[0, T]$ goes to zero almost surely. Our main tool will be the Burkholder-Davis-Gundy inequality, see [RY99, Ch. IV, Corollary 4.2] for details.

As the quadratic variation of the first summation in (12) is

$$Q_t := \sum_{i=1}^n \sum_{|v|=i} \int_0^t \left(\Gamma_s(v) - \Gamma_s^{(n)}(v) \right)^2 ds,$$

the BDG inequality gives us that for $h > 0$ there is a constant $C_h > 0$ such that

$$\mathbf{E} \left[\sup_{0 \leq t \leq T} \left| \sum_{i=1}^n \sum_{|v|=i} \int_0^t \left(\Gamma_s(v) - \Gamma_s^{(n)}(v) \right) dB_s(v) \right|^h \right] \leq C_h \mathbf{E} [Q_t^{h/2}].$$

Choose $h \leq 2$ so that, by subadditivity and a supremum bound on the integral terms, the right hand side is bounded above by

$$C_h T^{h/2} \sum_{i=1}^n \sum_{|v|=i} \mathbf{E} \left[\sup_{0 \leq t \leq T} |\Gamma_t(v) - \Gamma_t^{(n)}(v)|^h \right].$$

Now by choosing $h > 1$, Doob's maximal inequality gives that this is further bounded above by

$$C_h^* T^{h/2} \sum_{i=1}^n \sum_{|v|=i} \mathbf{E} \left[|\Gamma_T(v) - \Gamma_T^{(n)}(v)|^h \right].$$

By Lemma 2.6 the latter term goes to zero exponentially fast as $n \rightarrow \infty$, and then Borel-Cantelli completes the proof.

For the second summation of (12), the same argument with the BDG inequality yields that

$$\mathbf{E} \left[\sup_{0 \leq t \leq T} \left| \sum_{i=n+1}^{\infty} \sum_{|v|=i} \int_0^t \Gamma_s(v) dB_s(v) \right|^h \right] \leq C_h^* T^{h/2} \sum_{i=n+1}^{\infty} \sum_{|v|=i} \mathbf{E} \left[\Gamma_T(v)^h \right]. \quad (13)$$

From the proof of Lemma 3.3 we have that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \sum_{|v|=n} \mathbf{E} \left[\Gamma_T(v)^h \right] \leq \lambda_{\Gamma}(h) + \log \mathbf{E} \left[W_T^h \right] < 0$$

for h sufficiently close to 1, and hence by (13) and the Borel-Cantelli lemma the second summation of (12) term goes to zero almost surely. \square

Using equation (4) this leads to the following formulas for the evolution at other vertices:

Corollary 4.2. *For $v \in T$ the mass $\Gamma_t(v)$ evolves as*

$$\frac{d\Gamma_t(v)}{\Gamma_t(v)} = \sum_{i=1}^n dB_t(v_i) + \sum_{\substack{u \in T(v) \\ u \neq v}} \frac{\Gamma_t(u)}{\Gamma_t(v)} dB_t(u),$$

where $\varsigma = v_0, v_1, v_2, \dots, v_n = v$ are the vertices from the root to v . In particular this gives that if u is not a descendant of v or vice-versa then

$$\frac{d \langle \Gamma_t(u), \Gamma_t(v) \rangle}{\Gamma_t(u) \Gamma_t(v)} = |u \wedge v| dt,$$

where $u \wedge v$ is the last common ancestor of the paths to u and v .

Proposition 4.1 says that the total mass evolves as a continuous time exponential martingale. Its logarithm accumulates quadratic variation at a rate given by the last expression of Proposition 4.1, and, as is well known, the time at which an exponential martingale hits zero is equivalent to the time at which the accumulated quadratic variation reaches infinity. This gives another interpretation of the lifetime of the Γ_t process:

Corollary 4.3. *The Γ_t process reaches the zero measure at exactly the time*

$$\sup \left\{ t \geq 0 : \int_0^t \sum_{\substack{v \in T \\ v \neq \varsigma}} \left(\frac{\Gamma_s(v)}{\Gamma_s(\varsigma)} \right)^2 ds < \infty \right\}.$$

Before this time, that the total mass process is an exponential martingale naturally suggests the Girsanov theory plays a role here. This leads to the following:

Corollary 4.4. *Let P be the measure under which the vertex processes $\{B_t(v)\}_{v \in T}$ are independent Brownian motions. Assume that Γ is a probability measure. For any $T' < T$, let $\tilde{P}_{T'}$ be the probability measure whose Radon-Nikodym derivative with respect to P is $\Gamma_{T'}(\varsigma)$. Then under $\tilde{P}_{T'}$ the processes*

$$\left\{ t \mapsto B_t(v) - \int_0^t \Gamma_s(v) ds, 0 \leq t \leq T' \right\}_{v \in T}$$

are independent Brownian motions on the tree vertices.

See [RY99] for background on the Girsanov theory. In the next section we describe an application of this result to the model of tree polymers.

5 Applications to Other Models

5.1 Tree Polymers

Although this paper was written in the language of multiplicative cascades it was strongly motivated by the literature on tree polymers. The polymer model is virtually identical but the language is mildly different: to the vertices of the tree attach iid random variables $\{\omega(v)\}_{v \in T}$, and at inverse temperature β and level n define the polymer measure on ∂T by

$$d\Gamma_{\omega, \beta}^{(n)}(\xi) := \frac{1}{Z_{\omega, \beta}^{(n)}} \prod_{i=1}^n \exp \{ \beta \omega(\xi_i) \} d\Gamma(\xi).$$

Here $Z_{\omega, \beta}^{(n)}$ is the partition function

$$Z_{\omega, \beta}^{(n)} = \int_{\partial T} d\Gamma_{\omega, \beta}^{(n)}(\xi) = \Gamma_{\omega, \beta}^{(n)}(\rho).$$

In the tree polymer model we usually assume that Γ is a probability measure, and hence the partition function normalizes the polymer measure to also have mass one. Typically only the Lebesgue measure θ is used as the base measure, but we will continue to describe the model in this greater generality where any Γ can be used. The only assumption on the ω is that $e^{\lambda(\beta)} := \mathbf{E} [e^{\beta \omega}] < \infty$ for all $\beta \in \mathbb{R}$. Clearly then the polymer measure can be expressed as a cascade measure with

$$d\Gamma_{\omega, \beta}^{(n)}(\xi) = \frac{e^{n\lambda(\beta)}}{Z_{\omega, \beta}^{(n)}} d\Gamma_{W_\beta}^{(n)}(\xi) = \frac{d\Gamma_{W_\beta}^{(n)}(\xi)}{\Gamma_{W_\beta}^{(n)}(\varsigma)},$$

with $W_\beta(v) = \exp\{\beta\omega(v) - \lambda(\beta)\}$. If Γ is W_β -regular then Section 2 shows that the limiting polymer measure exists and is given by

$$\lim_{n \rightarrow \infty} d\Gamma_{\omega, \beta}^{(n)}(\xi) = \frac{d\Gamma_{W_\beta}(\xi)}{\Gamma_{W_\beta}(\zeta)}.$$

If Γ is not W_β -regular it is still an open problem as to whether or not a limit exists. Subsequential limits automatically exists because each finite level polymer measure is normalized to be a probability measure and the tree boundary ∂T is compact, but the structure of the set of subsequential limits is not known. See [WW10] for more on this problem.

Applying our cascade process to the study of polymer measures is most helpful whenever the family of cascading distributions $W_\beta = \exp\{\beta\omega - \lambda(\beta)\}$ can be represented by a process W_t satisfying Assumption 1. By this we mean that the processes W_β and W_t have the same marginal distributions at fixed times (up to a possible change of variables between β and t), but W_t has the independent increments property of Assumption 1. In this case, the cascade process of Section 3 gives us a coupling of the polymer measures at different temperatures that is different from the standard one obtained by simply multiplying the same variables by a different factor. The advantage of our coupling is that it has the Markov property implied by Section 3.3. In polymer language this Markov property has a nice interpretation: the polymer measure at a given temperature can be constructed by choosing a polymer at any higher temperature and then placing it in a new and independent environment. Most importantly, the higher temperature does *not* have to be infinite.

The simplest case of a weight process satisfying the above is the Gaussian weights of Section 4. The scaling properties of Brownian motion imply that in this case the t variable acts as both a time and an inverse temperature. This gives a nice interpretation to the stochastic calculus results of Proposition 4.1. The SDE for $\Gamma_t(\zeta)$ tells us that the total mass at the root evolves according to a weighted measure of the Brownian noise being inputted, with the weights prescribed by the polymer measure at the time infinitesimally beforehand. The formula for the quadratic variation tells us that it evolves according to the *overlap* of the polymer measure, that is the expected amount of time that two polymers paths chosen independently under Γ_t^* will spend together before eventually splitting. The explosion time of the cascade process is exactly when the accumulated overlap reaches infinity.

The Girsanov theory is also useful in this context. The tree polymer model can be thought of as a model of random walk in a random environment, where the random variables ω act as the environment. For this part we assume that $\Gamma = \theta$, and under the measure $\theta_{W_\beta}^*$ the process $\xi_0, \xi_1, \xi_2, \dots$, is Markov with transition probabilities given by

$$\theta_{W_\beta}^*(\xi_{i+1} = (\xi_i)_L | \xi_0, \xi_1, \dots, \xi_i) = \frac{\theta_{W_\beta}((\xi_i)_L)}{\theta_{W_\beta}(\xi_i)}.$$

To study this type of RWRE one typically uses the “point of view of the particle”, which is the study of the environment Markov chain defined by

$$Z_n = \{\omega(u)\}_{u \in T(\xi_n)}.$$

Note that Z_n takes values in the space of environments. It is straightforward to verify that if Q is a measure under which the ω are iid random variables and ξ is chosen according to the polymer measure $\theta_{W_\beta}^*$, then Z_n is a stationary Markov process with the same transition probabilities as the ξ_i Markov chain, i.e.

$$P(Z_{i+1} = \{\omega(u)\}_{u \in T((\xi_i)_L)} | Z_0, \dots, Z_i) = \theta_{W_\beta}^*(\xi_{i+1} = (\xi_i)_L | \xi_0, \xi_1, \dots, \xi_i).$$

See [Zei04] for more on the environment Markov chain. It begins in stationarity, with the stationary distribution being $\theta_{W_\beta}(\varsigma) dQ(\omega)$. The Girsanov theory of Corollary 4.4 gives a way to analyze this stationary distribution. Assume that under Q the ω are iid $N(0, T')$ for some $T' < 2 \log 2$. Then under $\theta_\omega(\varsigma) dQ(\omega)$ the variables ω have the law of

$$\int_0^{T'} \frac{\theta_s(v)}{\theta_s(\varsigma)} ds + \tilde{B}_{T'}(v),$$

where the $\tilde{B}_t(v)$ are iid Brownian motions on the vertices of the tree. This gives an alternate description of the stationary measure for the environment Markov Chain.

5.2 One-Dimensional Random Geometry and KPZ

Multiplicative cascades have also been used as a toy model for studies of random geometry, most notably in [BS09]. There one considers the pushforward of Γ_W onto the interval $[0, 1]$ via binary expansion; left turns in ξ correspond to zeros in the binary expansion and right turns to ones. We use Γ_W to also denote the distribution function of the measure on $[0, 1]$, i.e.

$$\Gamma_W(x) = \Gamma_W([0, x]).$$

If Γ_W is strictly positive, then $\Gamma_W(x)$ is a continuous, non-decreasing function on $[0, 1]$. If $\Gamma_W(v) > 0$ for every $v \in T$, then $x \mapsto \Gamma_W(x)$ is strictly increasing and hence a continuous bijection of $[0, 1]$ onto $[0, \Gamma_W(1)]$. In the case $\Gamma = \theta$, Benjamini and Schramm used this map to establish a relation between the Hausdorff dimension of a set and its random image under θ_W . Specifically they show the following:

Theorem 5.1 ([BS09]). *Let W be a cascading distribution with $\mathbf{E}[W \log W] < \log 2$ (so that θ is W -regular), and assume that $\mathbf{E}[W^{-s}] < \infty$ for all $s \in [0, 1]$. Let $K \subset [0, 1]$ be some non-empty, deterministic set. Then there is the following **KPZ formula**:*

$$\dim_H K = \phi_W(\dim_H \theta_W(K)),$$

where $\theta_W(K)$ is the (random) image of K via the distribution function θ_W , and ϕ_W is the deterministic bijection from $[0, 1]$ onto $[0, 1]$ given by

$$\phi_W(h) = h - \log_2 \mathbf{E}[W^h].$$

Applying our process to this setup gives some interesting interpretations. Let θ_t and ϕ_t denote the corresponding cascade process and bijection when we replace W by dynamic weights W_t . As time evolves, the image set $\theta_t(K)$ moves about the line and its Hausdorff changes with it, yet the dimension evolves deterministically even though the set evolves randomly. Remark 6 and the formula above tell us that $\phi_t(h)$ is a decreasing function of t for each fixed h , and hence Hausdorff dimensions get smaller as time evolves. Using our process it is possible to understand the infinitesimal evolution of the dimension. Indeed write $d(t) = \dim_H \theta_t(K)$, and then the KPZ formula becomes

$$d(0) = \phi_t(d(t)).$$

Differentiating both sides with respect to t leads to an ODE for $d(t)$:

$$\dot{d}(t) = -\frac{\dot{\phi}_t(d(t))}{\phi'_t(d(t))}.$$

The particulars of this ODE depends on the type of weight process being used. For example in the case of Gaussian weights as in Section 4 it becomes

$$\dot{d} = -\frac{d(1-d)}{2\log 2 - t(2d-1)} =: \psi_t(d).$$

This ODE has many interesting aspects. First note that the $2\log 2$ appears because it is the lifetime of the θ_t process, that is the time at which it collapses to the zero measure. Further, by the presence of the t term in the denominator the ODE is non-autonomous, except at $d = 1/2$ where the non-autonomous term strangely disappears. It can also be shown that

$$\lim_{t \uparrow 2\log 2} d = 1 - \sqrt{1 - d(0)},$$

so that even as θ_t approaches the zero measure the Hausdorff dimension of the random set stays bounded away from zero.

Although the work of Benjamini and Schramm can be used to derive the infinitesimal evolution of the Hausdorff dimension, in principle it should be possible to derive it separately and use it to give an alternate proof of their KPZ formula. All that needs to be found is a proof of the relation

$$\dim_H \theta_{t+\delta}(K) = \dim_H \theta_t(K) + \psi_t(\dim_H \theta_t(K))\delta + o(\delta)$$

that does not use the Benjamini and Schramm statement (although many of the techniques of their proof would probably be incorporated), and then the Markov property of the θ_t process turns this infinitesimal relation at a fixed time into the ODE that holds at all times. We have attempted to derive this relation but thus far been unable to, although we hope a proof will be at hand soon. In fact we believe that there is a slightly more general fact lurking in the background: namely that if Γ is an initial measure and W a cascading distribution that is a small perturbation away from the degenerate distribution at one, then

$$\dim_H \Gamma_W(K) = \dim_H \Gamma(K) + \psi_{\Gamma,W}(\dim_H \Gamma(K)).$$

Here $\psi_{\Gamma,W}$ would be a deterministic function determined by the properties of Γ and the size and type of the perturbation of W away from one. The infinitesimal relation is given by the “derivative” of ψ as the cascading distribution concentrates at one. It is not clear to us exactly how the properties of Γ enter into the picture, although we expect that they must in some form. It is also not clear if the relation above will be independent of the set K for all initial measures Γ , although we expect it will be for initial measures with some type of self-similarity.

References

- [Big77] J. D. Biggins. Martingale convergence in the branching random walk. *J. Appl. Probability*, 14(1):25–37, 1977.
- [BKL02] Anton Bovier, Irina Kurkova, and Matthias Löwe. Fluctuations of the free energy in the REM and the p -spin SK models. *Ann. Probab.*, 30(2):605–651, 2002.
- [BS09] Itai Benjamini and Oded Schramm. KPZ in one dimensional random geometry of multiplicative cascades. *Comm. Math. Phys.*, 289(2):653–662, 2009.
- [CN95] F. Comets and J. Neveu. The Sherrington-Kirkpatrick model of spin glasses and stochastic calculus: the high temperature case. *Comm. Math. Phys.*, 166(3):549–564, 1995.

- [EK86] Stewart N. Ethier and Thomas G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, 1986. Characterization and convergence.
- [Fan02] Ai Hua Fan. On Markov-Mandelbrot martingales. *J. Math. Pures Appl. (9)*, 81(10):967–982, 2002.
- [HW92] Richard Holley and Edward C. Waymire. Multifractal dimensions and scaling exponents for strongly bounded random cascades. *Ann. Appl. Probab.*, 2(4):819–845, 1992.
- [KP76] J.-P. Kahane and J. Peyrière. Sur certaines martingales de Benoit Mandelbrot. *Advances in Math.*, 22(2):131–145, 1976.
- [LR00] Quansheng Liu and Alain Rouault. Limit theorems for Mandelbrot’s multiplicative cascades. *Ann. Appl. Probab.*, 10(1):218–239, 2000.
- [MCRT11] David Márquez-Carreras, Carles Rovira, and Samy Tindel. A model of continuous time polymer on the lattice. *Commun. Stoch. Anal.*, 5(1):103–120, 2011.
- [OW00] Mina Ossiander and Edward C. Waymire. Statistical estimation for multiplicative cascades. *Ann. Statist.*, 28(6):1533–1560, 2000.
- [RY99] Daniel Revuz and Marc Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.
- [Tin05] Samy Tindel. On the stochastic calculus method for spins systems. *Ann. Probab.*, 33(2):561–581, 2005.
- [vBE65] Bengt von Bahr and Carl-Gustav Esseen. Inequalities for the r th absolute moment of a sum of random variables, $1 \leq r \leq 2$. *Ann. Math. Statist.*, 36:299–303, 1965.
- [WW10] Edward C. Waymire and Stanley C. Williams. T-martingales, size biasing, and tree polymer cascades. In *Recent developments in fractals and related fields*, Appl. Numer. Harmon. Anal., pages 353–380. Birkhäuser Boston Inc., Boston, MA, 2010.
- [Zei04] Ofer Zeitouni. Random walks in random environment. In *Lectures on probability theory and statistics*, volume 1837 of *Lecture Notes in Math.*, pages 189–312. Springer, Berlin, 2004.